Tutorial 3 (30 Sep)

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Q1) (a) Show that for any $0 < \alpha \leq \frac{1}{2}$, $x \in [-\pi, \pi]$, $\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$ converges.

(b) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$ is NOT a Fourier series of any 2π-periodic integrable function

Sol) (a) Idea: Apply Dirichlet Test for convergence of series.

(Almost the same proof as in Tutorial 1, Q2b: to be reproduced below)

Recall Dirichlet test (See e.g. Bartle's "Introduction to Real Analysis", 9.3.4):

Prop If the sequences of real numbers $\{x_n\}$, $\{y_n\}$ satisfy

- $\{x_n\}$: decreasing with $\lim_{n\to\infty} x_n = 0$.
- $\left\{\sum_{n=1}^{m} y_n\right\}_{m=1}^{\infty}$: bounded.

then $\lim_{m\to\infty} \sum_{n=1}^{m} x_n y_n$ exists. — $\square$

$\forall x \in [-\pi, \pi]$, apply the test with $x_n = \frac{1}{n^{\alpha}}$, $y_n = \sin nx$.

- $\left\{\frac{1}{n^{\alpha}}\right\}$: decreasing with $\lim_{n\to\infty} \frac{1}{n^{\alpha}} = 0$.
- For $x = 0, \pm\pi$, $\sum_{n=1}^{m} \sin nx = 0$; For $x \neq 0$, $\sum_{n=1}^{m} \sin nx = \operatorname{Im}\left(\sum_{n=1}^{m} e^{inx}\right) = \operatorname{Im}\left(e^{ix}\left(\frac{1-e^{imx}}{1-e^{ix}}\right)\right)$

$$\Rightarrow \left|\sum_{n=1}^{m} \sin nx\right| \leq \left|e^{ix}\left(\frac{1-e^{imx}}{1-e^{ix}}\right)\right| \leq 1 \cdot \frac{1+1}{|1-e^{ix}|} = \frac{2}{|1-e^{ix}|}$$ is bounded.

$\therefore \sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$ converges pointwisely on $[-\pi, \pi]$.

(b) Idea: Prove by contradiction using Parseval Identity.

Suppose on the contrary, there exists 2π-periodic integrable function f such that

$$S(f) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$$

then by Parseval Identity, $\pi \cdot \sum_{n=1}^{\infty} \left(\frac{1}{n^{\alpha}}\right)^2 = \|f\|_2^2 < +\infty$.

However, for $0 < \alpha \leq \frac{1}{2}$. $2\alpha \leq 1$, hence $\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}}$ diverges.

This leads to contradiction.

Therefore, $\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$ is not a Fourier series of any 2π-periodic integrable function

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Rmk. Actually, the same conclusion holds for $\frac{1}{2} < \alpha < 1$.

However, argument in (b) cannot carry through: $\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}}$ converges.

Please refer to e.g. [Stein: Ch. 3, Problem 1] for the (more difficult) argument.

- Note that by Tutorial 1, Q2a, for $\alpha = 1$, $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is the Fourier series of

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & -\pi \leq x < 0 \\ 0, & x = 0 \\ \frac{\pi}{2} - \frac{x}{2}, & 0 < x \leq \pi \end{cases}$$

- For $\alpha > 1$, $\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\alpha}}$ converges uniformly on $[-\pi, \pi]$, hence is the Fourier series of itself

Q2) Let $h: \mathbb{R} \to \mathbb{C}$ be a C^1 function such that h is of moderate decrease :

there exists $A, M > 0$ such that for all $|x| \geq A$, $|h(x)| \leq \frac{M}{1+x^2}$

(a) Show that $\hat{h}: \mathbb{R} \to \mathbb{C}$ defined by $\hat{h}(\xi) := \int_{-\infty}^{+\infty} h(x) e^{-2\pi i \xi x} dx$ is well-defined.

(b) Show that $h(x) := e^{-\pi x^2}$ is of moderate decrease with $\hat{h}(\xi) = e^{-\pi \xi^2}$.

Sol) (a) Idea: Use assumption to control the asymptotic behaviour of h.

Recall that $\int_{-\infty}^{+\infty} h(x) e^{-2\pi i \xi x} dx$ is well-defined $\Leftrightarrow$ $\lim_{K \to \infty} \int_{-K}^{K} h(x) e^{-2\pi i \xi x} dx$ exists

(Cauchy Criterion for improper integrals)

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists N > 0$ such that $\forall K > L > N$, $\begin{cases} \left|\int_L^K h(x) e^{-2\pi i \xi x} dx\right| < \varepsilon \\ \left|\int_{-K}^{-L} h(x) e^{-2\pi i \xi x} dx\right| < \varepsilon \end{cases}$

Given $\varepsilon > 0$, Choose $N > A$, to be determined later, then $\forall K > L > N$,

$$\left|\int_L^K h(x) e^{-2\pi i \xi x} dx\right| \leq \int_L^K \frac{M}{1+x^2} dx = M\left[\tan^{-1}(x)\right]_L^K = M(\tan^{-1}K - \tan^{-1}L) < \varepsilon$$

by choosing N large enough such that $\forall K > L > N$, $|\tan^{-1}K - \tan^{-1}L| < \frac{\varepsilon}{M}$.

in which such N exists, $\because$ $\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2} < +\infty$ and apply Cauchy Criterion.

Similarly for $\left|\int_{-K}^{-L} h(x) e^{-2\pi i \xi x} dx\right| < \varepsilon$, using $\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2} < +\infty$ instead.

$\therefore$ $\hat{h}(\xi)$ is well-defined.

(b) Showing $h(x)=e^{-\pi x^2}$ is of moderate decrease : note that $\lim_{x\to\pm\infty}(1+x^2)e^{-\pi x^2}=0$.

$\therefore$ Choose $M=1$, there exists $A>0$ such that $\forall\ |x|\geq A$,

$$\left|(1+x^2)e^{-\pi x^2}\right|\leq M \text{ , i.e. } |h(x)|\leq \frac{M}{1+x^2}$$

$\therefore$ $h(x)$ is of moderate decrease.

Showing $\hat{h}(\xi)=e^{-\pi\xi^2}$: Consider $g(\xi):=\hat{h}(\xi)\,e^{\pi\xi^2}$; it suffices to show that

① $g'(\xi)\equiv 0$ and ② $g(0)=1$

① $g'(\xi)=\left(\frac{d}{d\xi}\left(\int_{-\infty}^{+\infty}h(x)\,e^{-2\pi i\xi x}dx\right)+\hat{h}(\xi)\cdot(2\pi\xi)\right)e^{\pi\xi^2}$

$=\left(-2\pi i\int_{-\infty}^{+\infty}x\cdot h(x)e^{-2\pi i\xi x}dx+\hat{h}(\xi)\cdot(2\pi\xi)\right)e^{\pi\xi^2}$ (By differentiation under integral sign)

$=\left(-2\pi i\int_{-\infty}^{+\infty}-\frac{1}{2\pi}h'(x)\,e^{-2\pi i\xi x}dx+\hat{h}(\xi)\cdot(2\pi\xi)\right)e^{\pi\xi^2}$ ($\because h'(x)=-2\pi x\,e^{-\pi x^2}=-2\pi x\,h(x)$)

$=\left(i\left(\left[h(x)e^{-2\pi i\xi x}\right]_{x=-\infty}^{+\infty}-\int_{-\infty}^{+\infty}h(x)\cdot d\left(e^{-2\pi i\xi x}\right)\right)+\hat{h}(\xi)\cdot(2\pi\xi)\right)e^{\pi\xi^2}$ (By integration by part)

$=\left(i\left(-\int_{-\infty}^{+\infty}h(x)\cdot(-2\pi i\xi)\,e^{-2\pi i\xi x}dx\right)+\hat{h}(\xi)\cdot(2\pi\xi)\right)e^{\pi\xi^2}$ ($\because$ h is of moderate decrease)

$=\left((-2\pi\xi)\,\hat{h}(\xi)+\hat{h}(\xi)\cdot(2\pi\xi)\right)e^{\pi\xi^2}=0$

② $g(0)=\hat{h}(0)=\int_{-\infty}^{+\infty}e^{-\pi x^2}dx=1$ (By considering $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-\pi(x^2+y^2)}dx\,dy$)

$\therefore$ $g(\xi)\equiv 1$, i.e. $\hat{h}(\xi)=e^{-\pi\xi^2}$.

Rmk · In (a), the map $h(x) \longmapsto \hat{h}(\xi)$ is called the Fourier Transform.

· For integrable $f: \mathbb{R} \to \mathbb{C}$ supported on $[-1,1]$ (i.e. $f \equiv 0$ outside $[-1,1]$)

for any $k \in \mathbb{Z}$, $c_k(f) = \hat{f}(k)$. $\therefore$ Fourier Transform generalises

$f \longmapsto (c_k(f))_{k=-\infty}^{+\infty}$ to non-compactly-supported f and allowing $k \notin \mathbb{Z}$.

· Argument in (b) implies $h(x) = e^{-\pi x^2} = \int_{-\infty}^{+\infty} e^{-\pi\xi^2} e^{2\pi i x\xi} \, d\xi = \int_{-\infty}^{+\infty} \hat{h}(\xi) e^{2\pi i \xi x} \, d\xi$.

known as the Fourier Inversion formula, which actually holds for general h

as a "continuous analogue" of $f(x) \sim \sum_{k=-\infty}^{+\infty} c_k(f) e^{2\pi i k x}$ for f: 1-periodic.